Free Transport Limit for N-particles Dynamics with Singular and Short Range Potential

J. Barré · P.E. Jabin

Abstract We study the limit of systems of interacting particles, when the number of particle becomes very large. The support of the interaction vanishes as the number of particles goes to infinity, so that the natural limit is just free transport, but no limitation is assumed about the strength of the interaction. We obtain explicit estimates for the number of particles effectively interacting and describe the way they do it.

Keywords Kinetic theory of gases · Interacting particles systems

1 Introduction

We study the dynamics of many interacting particles in the limit of an infinite number of particles. The force acting on each particle is a sum of pair-wise interaction with the other particles. The kind of dynamics that is expected at the limit depends on the scaling of the force term with respect to the number of particles N. Here we consider very short range interaction in the sense that the force between two particles vanishes if their distance is larger than R with, in dimension 3

$$NR^2 \ll 1.$$

In this scaling the formal limit is simply free transport: each particle moves with its initial velocity. Indeed a formal computation easily shows that on average a particle should never undergo a collision: *i.e.* the number of particles coming at a distance less than R to another is negligible in front of N, in a time interval of order 1.

This is a simplified problem for the more interesting case $NR^2 = const$ where the obtention of collisional models (of Boltzmann type) is conjectured. The first rigorous step in that direction was obtained by Lanford in a celebrated work [10] (see also [11]). This result nevertheless suffers from two important restrictions: first the particles are hard spheres (they

J. Barré (✉) · P.E. Jabin
Laboratoire J. A. Dieudonné, UMR CNRS 6621, Université de Nice-Sophia Antipolis, Parc Valrose, 06108 Nice, France
e-mail: jbarre@math.unice.fr

interact with the potential $\Phi(x) = +\infty$ if $|x| \leq R$ and $\Phi = 0$ if $|x| > R$). And second the limit is only valid for a time small with respect to the average time it takes for one particle to have a collision (or the mean free path after rescaling).

This second restriction was improved by Cercignani, Illner and Pulvirenti in [3] (see also [8]); the time interval of validity was still finite and of the order of the mean free path though. Both results deal with hard spheres although extensions to repulsive potentials are mentioned (see [11]) but unpublished (as far as we know).

Those results are easily extended to our scaling. The limitation on the time interval is no more a real issue as the mean free path tends to infinity. However the proofs would still require hard spheres interaction (or at least a repulsive potential). Note that in the case of an arbitrary potential, the velocity bounds needed to substantiate the intuition behind the formal scaling argument are not easy to prove; in addition, the very concept of collision needs a careful definition. As a consequence, we do not know how to generalize Lanford's results to a general potential, even in the shorter range scaling considered here.

We start with Proposition 2.4 which justifies the free transport limit, still only for repulsive potentials. The interest is mainly in the simplicity of the proof as the result is not really new (although strictly speaking never stated before for a potential with so little regularity).

The main result is Theorem 2.5. It unfortunately deals with an even shorter range $R \ll N^{-3/5}$ but it assumes virtually nothing on the potential (it does not need to be repulsive and its scale could be as large as one wants) and therefore it is not at all included in previous contributions. Moreover we not only prove the free transport limit but also describe all possible collision sequences between particles and the result is consequently much more precise. The core of the proof is to ensure that the build up of correlations between particles does not destroy the validity of simple scaling arguments. Theorem 2.5 leaves open the question of a non trivial limit in the range $N^{-1/2} \ll R < N^{-3/5}$ (for which we have no particular clue to offer). The particles velocities could then become unbounded and the limit would not necessarily be free transport (as the probability of interacting with another particle would increase accordingly).

Finally and for the sake of completeness, let us mention that the limit dynamic is also an important and fertile field of study in the case of long range potentials (and obviously with a weaker scaling). Vlasov-like kinetic equations are then expected. The first rigorous results were obtained by Braun and Hepp [2], Neunzert and Wick [12] and Dobrushin [4], and concern very regular interaction potentials. We also refer to Spohn [15] where the Wasserstein distance is used to obtain the convergence. Two references related to numerical simulations with particles' methods are for instance Victory and Allen [16] and Wollman [17]. All these works require a regular interaction kernel (at least continuous and usually even lipschitz). Much less is known concerning irregular interactions: the convergence to the Euler equation of point vortex systems is well controlled [5, 13, 14], see also [9], and recently, Hauray and Jabin [7] proved the convergence of a particle approximation to the Vlasov equation for a class of weakly singular potentials. However, the physical potentials of interest are typically the much more singular Coulombian and gravitational interactions, for which no convergence results to the Vlasov equation are known. This difficulty to deal with the short range singularity of potentials in the Vlasov limit was also an incentive for the study of this paper: what happens for a general singular potential in a short range scaling.

2 Setting and Statement of the Results

We study the perturbation of the free transport of N point particles by a very short range (but strong) interaction potential. Denote respectively by $X_i(t) \in \mathbb{R}^3$ and by $V_i(t) \in \mathbb{R}^3$ the

position and velocity at time t of the particle number i. The X's and V's are governed by the following equations of motion:

$$\dot{X}_i = V_i,$$

$$\dot{V}_i = \frac{1}{S_N} \sum_{j \neq i} K_\varepsilon(X_i - X_j), \quad i = 1, \ldots, N, \tag{2.1}$$

where S_N is a scaling factor. $S_N = N$ is the usual Vlasov scaling; in the following, S_N will be N or 1. $K_\varepsilon(x)$ is compactly supported in the ball of radius ε^k but singular. It is assumed to be derived from a potential

$$K_\varepsilon = -\nabla_x \Phi_\varepsilon, \quad K_\varepsilon(-x) = -K_\varepsilon(x). \tag{2.2}$$

This is not strictly necessary and could be replaced by a weaker assumption on the compressibility and energy bounds for the corresponding flow but is generally satisfied by the physical kernels we would wish to use (with some exceptions).

The parameter ε is directly connected to the number of particles, being the average distance in the phase space between two particles or

$$\varepsilon = N^{-1/6}.$$

The initial conditions X_i^0 and V_i^0 are typically random although very large values should be avoided. For simplicity we will consider initial values $I = (X_1^0, V_1^0, \ldots, X_N^0, V_N^0)$ in the hypercube $I \in [-1, 1]^{6N}$. In the following, the measure of a set A will always mean the Lebesgue measure, and be denoted by $|A|$.

Our aim is to compare this dynamics to the unperturbed one, which is simply given by

$$\frac{d\bar{X}_i}{dt} = V_i, \quad \frac{d\bar{V}_i}{dt} = 0, \quad i = 1, \ldots, N. \tag{2.3}$$

If $S_N = N$, this is therefore a kind of mean field limit (but in a weak form) for which much is known provided K_ε is regular enough. Here we only assume the minimal regularity necessary to ensure that the system (2.1) is well defined, that is (see Ambrosio [1] or Hauray for the particular case considered here [6])

$$K_\varepsilon \in BV(\mathbb{R}^3), \quad \text{supp} K_\varepsilon \subset B(0, \varepsilon^k). \tag{2.4}$$

Note that in addition any other assumption ensuring well posedness of (2.1) for a given N would be enough.

Let us first give some definitions.

Definition 2.1 Two particles are said to collide iff

$$\exists t \quad \text{s.t.} \quad |X_i(t) - X_j(t)| \leq \varepsilon^k. \tag{2.5}$$

Let us note that we have no information on the duration of a collision; unless otherwise stated, the "time of a collision" is to be understood in the following as the beginning time of the collision, that is the smallest time t such that $|X_i(t) - X_j(t)| \leq \varepsilon^k$.

Definition 2.2 We denote by $i R_T j$ the relation "i and j collide before time T".

Definition 2.3 A sequence of collisions is an orbit of the relation R_T.

An easy formal computation tells us that the average number of particles having a collision should be of order $N^2 \varepsilon^{2k}$. This number is large (if $k < 6$) but nevertheless small with respect to N if $k > 3$. If this can be rigorously justified, we would therefore obtain, at the limit $N \to \infty$, the dynamics (2.3) for almost all particles. As all particles are identical, proving this is equivalent to controlling the set of initial conditions for which a chosen particle (the first one for example) has a collision, or

Proposition 2.4 *Take* $S_N = N$. *Assume that* $\Phi_\varepsilon \geq 0$ *and* $|\Phi_\varepsilon| \leq C/|x|$. *For any* $6 > k > 3$ *(such that* $N\varepsilon^{2k} << 1$*), fix a time* T, N *large enough and consider* (2.1) *until time* T. *Then the set of initial data* I *for which the first particle has at least one collision is of measure less than* $CN\varepsilon^{2k}$.

Remark

The two big restrictions are the conditions that the potential be repulsive and that $S_N = N$. At first glance and formally, it would seem that they should not be necessary. They are however needed to control the velocities of the particles, in the cases of collisions involving 3 or more particles. Contrary to the case of Theorem 2.5 below, here 3-particles collision (or more) can and will typically occur. And notice that the formal computation predicting that only $N^2 \varepsilon^{2k}$ particles will undergo a collision, only holds if the velocities of the particles remain of order 1.

This proposition is enough to obtain the limit but it does not describe at all what happens to the particles which have collisions (except that they do not interact with too many other particles) or in the case of attractive (or non repulsive) potentials. If k is large enough, it is possible to be more specific in the sense that two particles may collide only once and a sequence of collisions is exactly that: collisions happening one after the other without overlapping. To be that specific it is unfortunately necessary to exclude the possibility of 3-particles collision. This is automatic for hard spheres, but is requires a more stringent condition if general potentials are included, namely $k > 18/5$.

More precisely

Theorem 2.5 *Take* $S_N = 1$ *and any* $k > 3$ *(such that* $N\varepsilon^{2k} \ll 1$*). Fix a time* T, N *large enough and consider* (2.1) *until time* T. *Then, there exists a large set* $O \subset [-1, 1]^{6N}$ *with*

$$|[-1, 1]^{6N} \setminus O| \leq C_T N^3 \varepsilon^{5k},$$

such that for all $I \in O$, *we have the following*

(i) *All collisions involve only two particles; that is*:

$$\forall i, j, l, \ \forall t < T, |X_i(t) - X_j(t)| > \varepsilon^k \quad or \ |X_i(t) - X_l(t)| > \varepsilon^k.$$

   This allows us to define the number of collisions in a given sequence.
(ii) *Two particles belonging to the same sequence at a certain time* $t < T$ *do not collide again before time* T.
(iii) *Define* O_n *as the subset of* O *containing the initial data for which there are at least one sequence with exactly* n *collisions in the dynamics. Then*

$$|O_n| \leq C_{T,n} N^{n+1} \varepsilon^{2kn}.$$

Remarks

(1) Point (ii) may be formulated more precisely: consider a collision sequence, and two indices i and j in this sequence undergoing an actual collision, that is $i R_T j$. Consider now the relation $R_{T,ij}$, defined as the relation R_T excluding $i R_T j$, and define A_i and A_j the orbits of i and j under $R_{T,ij}$. Then $A_i \cap A_j = \emptyset$. This amounts to say that particles correlated in some way by belonging to the same collision sequence do not collide again.

(2) The theorem is interesting only for k such that $N^3 \varepsilon^{5k} \leq 1$ and n large enough such that we have $N^{n+1} \varepsilon^{2kn} < 1$. Note that for $k > 3$, there is always a n such that the above quantity vanishes to 0 with N (remember that $N \varepsilon^6 = 1$). The first condition however implies that $k \geq 18/5$.

Let us stress that those two results leave open all the range of cases $3 < k < 18/5$ for general potentials. Of course the assumptions on the potential in Theorem 2.5 are very weak (K_ε could even blow up with N) but we do not know how to proceed even with a precise scaling ($S_N = N$ for instance).

We have stated Proposition 2.4 and Theorem 2.5 for a fixed time $T = O(1)$. As a final remark, let us note that Proposition 2.4 remains valid for an N-dependent time scale T as long as $T \ll N^{k/3-1}$; the same is true for Theorem 2.5 as long as $T \ll N^{5/6(k-k_c)}$, with $k_c = 18/5$. These extensions are easy consequences of the proofs.

3 Proof of Proposition 2.4

We first need some control on the maximal number of interacting particles.

Denote by B_0 the set of initial data such that any two particles i and j satisfy initially that

$$|X_i(0) - X_j(0)| > \frac{1}{N}.$$

Clearly, as there is no dynamics yet

$$|[-1, 1]^{6N} \setminus B_0| \leq \frac{1}{N}. \tag{3.1}$$

We define a K-collision between two particles i and j if there exists t such that $|X_i(t) - X_j(t)| \leq K\varepsilon^k$.

Denote now by B_n^K the subset of B_0 of initial conditions such that there is no sequence of K-collisions involving $n + 1$ particles or more before time T and by $O_n^K = B_0 \setminus B_n^K$. We will first prove that

$$|O_n^K| \leq C_{n,K} N^{n+1} \varepsilon^{2kn}. \tag{3.2}$$

We start with a control on the velocity

Lemma 3.1 *For $K \geq 1$, $I \in B_n^K$, any $i = 1, \ldots, N$ and any $t \in [0, T]$, then we have that*

$$|V_i(t)| \leq 1 + Cn.$$

Proof As $I \in B_n^K$ with $K \geq 1$, the particle i interacts with at most n other particles $j_1, \ldots, j_n$. Denote $j_0 = i$. As $\Phi_\varepsilon \leq 0$, we hence obtain by energy conservation that

$$\sum_{k=0}^{n} |V_{j_k}(t)|^2 \leq \sum_{k=0}^{n} |V_{j_k}(0)|^2 + \frac{1}{N} \sum_{0 \leq k \neq m \leq n} \Phi_\varepsilon(X_{j_k}(0) - X_{j_m}(0)).$$

Recalling that $I \in B_0$, we know that $|X_{j_k}(0) - X_{j_m}(0)| > 1/N$ and as $|\Phi_\varepsilon| \leq C/|x|$, we deduce that

$$\sum_{k=0}^{n} |V_{j_k}(t)|^2 \leq 1 + n + Cn^2,$$

hence the result.   □

Let us now prove by induction the estimate on $|O_n^K|$. The initial step and the $n \to n+1$ step are very similar so we do only the second one. Assume therefore that

$$|O_n^K| \leq C_{n,K} N^{n+1} \varepsilon^{2kn},$$

and consider a initial data $I \in O_{n+1}^K$. Necessarily there exists $j_1, \ldots, j_{n+1}$ and a time $t_0 < T$ such that before t_0 there was no sequence of K-collisions involving more than $n+1$ particles and at t_0 at least another particle, denoted by i, starts a K-collision with one of the j_k.

Notice that there could be more than one particle having a collision at t_0, in which case we just choose anyone of them. Moreover the particle i may have had collisions with another set of particles before t_0 but with none of the j_k.

Divide the time interval $[0, T]$ into $M \leq C(1 + n)T\varepsilon^{-k}$ small intervals $[t_\alpha, t_{\alpha+1}]$ of size less than $\varepsilon^k/2(1 + Cn)$.

Through Lemma 3.1, we know that until time t_0, the velocity of every particle is less than $1 + Cn$. Therefore choosing $t_\alpha = \max\{t_\beta | t_\beta < t_0\}$, we know that at t_α, $X_i(t_\alpha)$ is at a distance less than $(1 + K)\varepsilon^k$ from one of the $j_1, \ldots, j_{n+1}$. For the same reason, as $j_1, \ldots, j_{n+1}$ form a sequence of K-collisions before t_0, they also form a sequence of $K + 1$-collisions before t_α.

Consequently denote by $O_n^K(i, j_1, \ldots, j_{n+1}, \alpha)$ the set of initial data $I \in B_0$ such that $j_1, \ldots, j_{n+1}$ form a sequence of $K + 1$-collisions before t_α, i has no K-collision with any of the j_k before t_α and for some $k = 1, \ldots, n + 1$, $|X_i(t_\alpha) - X_{j_k}(t_\alpha)| \leq (K + 1)\varepsilon^k$.

Controlling the measure of this set is easy because there is no interaction between particles $j_1, \ldots, j_{n+1}$ and the other particles. Choose first $X_{j_1}(0), V_{j_1}(0), \ldots, X_{j_{n+1}}(0), V_{j_{n+1}}(0)$. The trajectory of the j_k particles is then determined. The measure of the set of initial values $X_k(0), V_k(0), k \neq j_1, \ldots, j_{n+1}$ is then less than the measure of the set of initial condition such that at a given time X_i is in a given ball of radius $(K + 1)\varepsilon^k$. By Galilean invariance, this is less than

$$C(K + 1)^3 \varepsilon^{3k}.$$

Finally the measure of the set of $X_{j_1}(0), V_{j_1}(0), \ldots, X_{j_{n+1}}(0), V_{j_{n+1}}(0)$ such that they form a sequence of $K + 1$-collisions before t_α is simply

$$C_{n,K+1} \varepsilon^{2nk}.$$

Indeed from the induction assumption for the previous step n (but with $K + 1$ instead of K), the measure of the set of initial data such that $n + 1$ particles form a sequence of $K + 1$-collisions is less than

$$C_{n,K+1} N^{n+1} \varepsilon^{2nk}.$$

As this set is invariant under permutations on the indices of the particles (and as the measure of the set for which two sequences of collisions occur is trivially much lower), we obtain the claimed bound.

Finally

$$|O_n^K(i, j_1, \ldots, j_{n+1}, \alpha)| \leq C C_{n,K+1}(K+1)^3 \varepsilon^{(2n+3)k}.$$

As

$$O_{n+1}^K \subset \bigcup_{i,j_1,\ldots,j_n=1}^{N} \bigcup_{\alpha=1}^{M} O_n^K(i, j_1, \ldots, j_{n+1}, \alpha),$$

we obtain that

$$|O_{n+1}^K| \leq N^{n+2} M C C_{n,K+1}(K+1)^3 \varepsilon^{2(n+1)k+1} \leq C_{n+1,K} N^{n+2} \varepsilon^{2(n+1)k},$$

with

$$C_{n+1,K} = C C_{n,K+1}(K+1)^3(1+n)T.$$

This proves (3.2).

With (3.2), it is quite straightforward to show Proposition 2.4. Indeed choose n large enough such that

$$N^{n+1}\varepsilon^{2kn} \leq N\varepsilon^{2k}.$$

It is enough to take $n(k/3 - 1) \geq k/3$ so that n does not depend on N. Denote by D_1 the set of initial data such that the first particle has a collision before time T. Denote $\tilde{D}_1 = D_1 \cap B_n^1$. Controlling $\tilde{D}_1$ is enough as

$$|[-1, 1]^{2N} \setminus B_n^1| \leq |[-1, 1]^{2N} \setminus B_0| + |B_0 \cap O_n^1| \leq \frac{1}{N} + C_{n,T} N^{n+1} \varepsilon^{2kn}$$

$$\leq C_T N \varepsilon^{2k},$$

as $k < 6$ and n is from now on fixed.

For any $I \in \tilde{D}_1$, there exists t_0 such that the first particle has no collision until t_0 and a collision occurs with another particle $i \neq 1$ (possibly more than 1) at t_0. As $I \in B_n^1$, all velocities are bounded by $1 + Cn$. Consequently if we again divide $[0, T]$ into M intervals $[t_\alpha, t_{\alpha+1}]$ with $M \leq C(1+n)T\varepsilon^{-k}$, and if we choose $t_\alpha = \max\{t_\beta < t_0\}$ then $|X_1(t_\alpha) - X_i(t_\alpha)| \leq 2\varepsilon^k$.

Denote by $\tilde{D}_1(i, \alpha)$ the subset of B_n^1 such that the first particle has no collision before t_α and $|X_1(t_\alpha) - X_i(t_\alpha)| \leq 2\varepsilon^k$. We have that

$$\tilde{D}_1 \subset \bigcup_{i=1}^{N} \bigcup_{\alpha=1}^{M} \tilde{D}_1(i, \alpha).$$

As until t_α the trajectory of the first particle is a line, it is obvious that

$$|\tilde{D}_1(i, \alpha)| \leq C\varepsilon^{3k},$$

so that

$$|\tilde{D}_1| \leq C N M \varepsilon^{3k} = C N \varepsilon^{2k},$$

which finishes the proof of Proposition 2.4.

4 Proof of Theorem 2.5

The theorem is proved in two steps. First assuming point (i) and (ii) are known, we prove (iii). We end up with the proof for points (i) and (ii).

4.1 Proof of Point (iii) Knowing (i) and (ii)

In this subsection we assume we have a set O such that for any $I \in O$, (i) and (ii) of Theorem 2.5 are true.

Proposition 4.1 *Let $I \in O$ and A be a sequence with n collisions. Then $\text{card}(A) = n + 1$.*

Proof We argue by induction. For $n = 1$, $\text{card}(A) = 2$, since one collision involves two, and only two, particles.

We suppose now that the property is true for all $m < n$. Let i and j be the two particles involved in the last collision of the sequence A. As above, we define A_i and A_j the collision sequences of i and j when the collision between i and j is excluded. Then, using Point (ii) of Theorem 2.5, $A_i \cap A_j = \emptyset$, so that $\text{card}(A) = \text{card}(A_i) + \text{card}(A_j)$. Now, A_i and A_j are sequences with respectively m and $n - 1 - m$ collisions. Using the induction hypothesis, we get

$$\text{card}(A) = m + 1 + (n - 1 - m + 1) = n + 1.$$

The proof is completed. $\qquad\square$

Let us now study O_n the subset of O such that for I in O_n the dynamics has at least one sequence with exactly n collisions. Then

Proposition 4.2 *Let $n + 1$ particles follow the equations of motion*:

$$\left\{ \begin{array}{l} \dot{X}_i = V_i \\ \dot{V}_i = \frac{1}{N} \sum_{j=1, j \neq i}^{n+1} K_\varepsilon(X_i - X_j) \end{array} \right\}. \tag{4.1}$$

We denote by ω_n the measure in $\mathbb{R}^{6(n+1)}$ of the initial conditions such that the $n + 1$ particles belong to the same sequence. Then $|O_n| \leq C_N^{n+1} |\omega_n|$.

Proof The argument is very simple. Consider the $n + 1$ particles composing the sequence with n collisions in O_n. There are C_N^{n+1} possible choices of indices for those particles but once they are chosen they follow exactly (4.1). Indeed as they belong to a sequence with exactly n collisions, they do not come within ε^k of any other particle.

Their initial positions and velocities therefore belong to ω_n hence the result. $\qquad\square$

We may now concentrate on the evaluation of $|\omega_n|$. The heuristics behind the next proposition is the following: two particles collide at a given time if they are inside a volume of order ε^{3k}; otherwise stated, the measure of initial conditions leading to collision at a given time is of order ε^{3k}, since measure is preserved by the flow. Now, the measure of initial conditions leading to collision at *any* time is of order ε^{2k}. The following proposition states that all collisions in a sequence of n brings one such factor ε^{2k}.

Proposition 4.3 *Under the same hypothesis as for Proposition* 4.2,

$$|\omega_n| \le a_{n,T}\varepsilon^{2nk}. \tag{4.2}$$

Point (iii) of Theorem 2.5 immediately follows from this proposition and the previous one.

Three lemmas are necessary for the proof of Proposition 4.3

Lemma 4.4 *Take* $I \in O$. *Let* i *be a particle belonging to a sequence of n collisions. Then whenever* i *is not colliding with another particle,* $|V_i| \le \sqrt{n+1}$.

Proof Energy conservation during a collision between i and j implies that

$$|V_i'|^2 + |V_j'|^2 = |V_i|^2 + |V_j|^2,$$

where the primes refer to quantities after collision. This equality is valid when the collision is over. Initially, $|V_i|^2 \le 1$ for all i. Thus after one collision, $|V_i|^2 \le 2$ for all i, which proves the property for $n = 1$.

We assume that the property is true for all $m < n$ and argue by induction.

Consider now any collision in the sequence, involving particles i and j; we call m_i and m_j the number of particles involved in the collision sequences of i and j before the considered collision. Then $m_i + m_j \le n - 1$. After the collision

$$\begin{aligned}
|V_i'|^2 &\le |V_i|^2 + |V_j|^2 \\
&\le m_i + 1 + m_j + 1 \\
&\le n + 1,
\end{aligned}$$

using the induction hypothesis. □

Lemma 4.5 *Take* $I \in O$. *Let* i *be a particle belonging to a sequence of m collisions. Then during a time interval* $[T_1, T_2]$, i *travels a distance smaller than* $\sqrt{m+1}(T_2 - T_1) + m\varepsilon^k$.

Proof When i is not colliding, its velocity is bounded by $\sqrt{m+1}$, thanks to Lemma 4.4.

When i is colliding with another particle j, it's not possible to bound its velocity, but its position X_i can be controlled by tracking G, the center of mass of particles i and j: the velocity of the center of mass V_G is bounded by $\sqrt{m+1}$, and during the collision, $|X_i(t) - X_G(t)| \le \varepsilon^k/2$. Thus for a collision lasting τ, particle i travels at most a distance $\tau\sqrt{m+1} + \varepsilon^k$. Finally, adding the effects of at most m collisions, particle i between T_1 and T_2 travels at most $\sqrt{m+1}(T_2 - T_1) + m\varepsilon^k$. □

Lemma 4.6 *Let* Ω *be a bounded region in* $\mathbb{R}^6$. *Consider the dynamics of* $n + 1$ *particles given by* (4.1). *We denote by* $p_{n,T}^{i,\Omega}$ *the measure in* $\mathbb{R}^{6(n+1)}$ *of the initial conditions such that the* $n + 1$ *particles belong to the same sequence, and at time* T, $(X_i(T), V_i(T)) \in \Omega$. *Then* $p_{n,T}^{i,\Omega} \le |\omega_n||\Omega|$.

Proof This is an easy consequence of the Galilean invariance of the dynamics. In other words if $X_j, V_j, j = 1, \dots, n + 1$, satisfy (4.1) and belong to the same sequence then for any $x, v \in \mathbb{R}^3$, $X_j + x + vt$, $V_j + v$ satisfy again (4.1) and belong to the same sequence. □

Proof of Proposition 4.3 We divide the time interval $[0, T]$ in M small intervals of size $\Delta t = \varepsilon^k/(2\sqrt{n+1})$, and consider the times $t_l = l\varepsilon^k/(2\sqrt{n+1})$, with l an integer. Then

$$M = [2T\sqrt{n+1}\varepsilon^{-k}] + 1 \leq 3T\sqrt{n+1}\varepsilon^{-k},$$

for ε small enough. Then if particles i and j collide, there exists l such that

$$|X_i(t_l) - X_j(t_l)| \leq (n+2)\varepsilon^k$$

This is a direct consequence of Lemma 4.5.

We call $I_{ij,l}$ the set of initial conditions such that: n collisions happen; l is the smallest integer such that $|X_i(t_l) - X_j(t_l)| \leq (n+2)\varepsilon^k$; the last collision occurs between particles i and j. "Last" means here that at time t_l, all collisions of the sequence, except perhaps the $i - j$ one, have started and may be completed.

We now argue by induction.

Case $n = 1$ There are only two particles involved, 1 and 2, so that

$$\omega_n \subset \cup_l I_{12,l}.$$

Now, owing to the conservation of the measure by the flow, we may evaluate the measure of $I_{12,l}$ by evaluating the measure of its image by the flow at any time. We choose the time t_l. At this time X_1, V_1 and V_2 are undetermined, and X_2 belongs to the ball centered in X_1, with radius $3\varepsilon^k$. Thus $|I_{12,l}| = 36\pi\varepsilon^{3k}$. From here, we write

$$|\omega_n| \leq \sum_{l=1}^{M} 36\pi\varepsilon^{3k} \leq 108T\sqrt{3}\varepsilon^{2k}.$$

This proves the $n = 1$ case.

We now assume that the property is true for all $m < n$.

Then we have

$$|\omega_n| \leq \sum_{i,j,l} |I_{ij,l}|.$$

We evaluate the measure of $I_{ij,l}$ by examining the situation at $t = t_l$, using again measure conservation. We call $A_i = \{i, i_1, \ldots, i_m\}$ and $A_j = \{j, j_1, \ldots, j_{n-1-m}\}$ the collision sequences to which i and j respectively belong before the $i - j$ collision. A_i and A_j are two disjoints sets thanks to point (ii) of Theorem 2.5. We call $I_m^{(i)}$ the set of initial conditions on $\{i, i_1, \ldots, i_m\}$ such that the collision sequence A_i happens; at time t_l, particle i is at a certain position $X_i(t_l) = X_0$. We call $I_{n-1-m}^{(j),X_0}$ the set of initial conditions on $\{j, j_1, \ldots, j_{n-1-m}\}$ such that the collisions of sequence A_j happens and particle j is at t_l in the ball centered in X_0, of radius $(n+2)\varepsilon^k$; using Lemma 4.6 with $\Omega = B(X_0, 2\varepsilon^k) \times \mathbb{R}^3$, we know that

$$|I_{n-1-m}^{(j),X_0}| \leq |\omega_{n-1-m}| \frac{4(n+2)^3\pi}{3}\varepsilon^{3k}.$$

We then evaluate the measure of $I_{ij,l}$ by

$$|I_{ij,l}| \leq \sum_{m=0}^{n-1} C_{n-1}^m |I_m^{(i)}||I_{n-1-m}^{(j),X_0}|$$

$$\leq C \sum_{m=0}^{n-1} C_{n-1}^m \, |\omega_m||\omega_{n-1-m}|\varepsilon^{3k}$$

$$\leq C \sum_{m=0}^{n-1} C_{n-1}^m a_m a_{n-1-m}\varepsilon^{3k+2k(n-1)}.$$

The last inequality uses the induction assumption. Finally, we have

$$|\omega_n| \leq \sum_{i,j=1}^{n} \sum_{l=1}^{M} \sum_{m=0}^{n-1} C_{n-1}^m a_m a_{n-1-m}\varepsilon^{3k+2k(n-1)}$$

$$\leq cT\sqrt{n+1}\varepsilon^{-k}n^2 \sum_{m=0}^{n-1} C_{n-1}^m a_m a_{n-1-m}\varepsilon^{3k+2k(n-1)}$$

$$\leq a_n\varepsilon^{2kn},$$

with

$$a_n = cT\sqrt{n+1}n^2 \sum_{m=0}^{n-1} C_{n-1}^m a_m a_{n-1-m}. \tag{4.3}$$

Notice that the expression for a_n is far from optimal and that better bounds could be obtained, this is however enough to complete the proof. $\qquad\square$

4.2 Proof of Points (i) and (ii) of Theorem 2.5

Let us first note that both (i) and (ii) are true for $T = 0$: (ii) obviously there cannot be recollisions as there is no dynamic; (i) as the set of initial data for which at least three particles are in the same ball of radius ε^k has measure less than $CN^3\varepsilon^{6k}$.

Therefore for any $I \in O$ with $|[-1, 1]^{2N} \setminus O| \leq CN^3\varepsilon^{6k}$, points (i) and (ii) hold at time 0.

Now fix $T > 0$, and denote by B the set of initial conditions I such that either (i) or (ii) are not true until time T. We will show that

$$|B| \leq C_T N^3 \varepsilon^{5k}.$$

For any $I \in B$, there exists $t_0 > 0$ such that (i) and (ii) are true until t_0 and one of them (or both) false at t_0 (t_0 depends on I). This is because both are true at time 0 and because the dynamics is continuous in time.

Notice that all results of Sect. 4.1 are true until t_0. As $k > 3$, there exists n independent of N such that asymptotically in N

$$N^{n+1}\varepsilon^{2kn} \leq N^3\varepsilon^{5k}.$$

Therefore, it is enough to consider $\tilde{B} = B \setminus O_n$ as $|O_n| \leq C_T N^3\varepsilon^{5k}$ and for any $I \in \tilde{B}$ there are at most $n - 1$ collisions in the dynamics for any sequence of collisions.

In fact we can be a bit more precise and notice that Theorem 2.5 is useless and trivial when $N^3\varepsilon^{5k}$ is not less than 1. Recalling that $\varepsilon = N^{-1/6}$ this implies that it is enough to consider $k \geq 18/5$. Consequently, $n = 5$ and there are at most 4 collisions (in any sequence) in the dynamics we consider.

We have to study both cases where (i) (respectively (ii)) is contradicted at time t_0. Denote by $\tilde{B}_1 \subset \tilde{B}$ the set of initial conditions $I \in \tilde{B}$ such that there exist $t_0 \leq T$ and three particles i, j, k with $X_i(t_0)$, $X_j(t_0)$ and $X_k(t_0)$ in the same ball of size $2\varepsilon^k$. Clearly $\tilde{B}_1$ contains all initial conditions of $\tilde{B}$ contradicting (i) but is a bit larger (which will be useful for (ii)).

Accordingly denote by $\tilde{B}_2$ the set of initial conditions $I \in \tilde{B} \setminus \tilde{B}_1$ for which there exists t_0, and i and j such that for some $t < t_0$, $i R_t j$ and i and j collide at t_0 with t_0 the beginning of the collision; which means that $|X_i(t_0) - X_j(t_0)| = \varepsilon^k$ and $|X_i(t) - X_j(t)| > \varepsilon^k$ for any $t < t_0$ but close enough from t_0. $\tilde{B}_2$ contains all initial conditions contradicting (ii) before (i) and of course $\tilde{B} = \tilde{B}_1 \cup \tilde{B}_2$.

4.2.1 Case of $\tilde{B}_1$: (i) Is Contradicted First

Denote by $\tilde{B}_1(t_1)$ the set of initial conditions I such that $I \in \tilde{B}_1$ and at t_1 we have three particles located in the same ball of size $3\varepsilon^k$. As there are C_N^3 possibilities of choosing those three particles, the measure of the set of $X_1(t_1), V_1(t_1), \ldots, X_N(t_1), V_N(t_1)$ such that three particles are located in the same ball of size $3\varepsilon^k$ at t_1 is less than

$$C N^3 \varepsilon^{6k}.$$

Therefore, since the dynamics preserves volume in the whole phase space, the set $\tilde{B}_1(t_1)$ has measure less than $C N^3 \varepsilon^{6k}$.

We divide the time interval $[0, T]$ into M intervals $[T_\alpha, T_{\alpha+1}]$, $\alpha = 0, \ldots, M$, of size $\varepsilon^k (n + 1)^{-1/2}$. If three particles are located in a ball of radius ε^k at t_0, Lemma 4.5 ensures that they are still within a ball of radius $6\varepsilon^k$ at some well chosen time T_α. We have also used that there at most 4 collisions in any sequence before t_0. So

$$\tilde{B}_1 = \bigcup_{\alpha=1}^{M} \tilde{B}_1(T_\alpha),$$

and as M is less than $T\varepsilon^{-k}\sqrt{n+1}$, we finally obtained that

$$|\tilde{B}_1| \leq \sum_{\alpha=1}^{M} |\tilde{B}_1(T_\alpha)| \leq C M N^3 \varepsilon^{6k} \leq C_T N^3 \varepsilon^{5k}, \tag{4.4}$$

which is the desired estimate on $\tilde{B}_1$.

4.2.2 Case of $\tilde{B}_2$: (ii) Is Contradicted First

Let us first perform the same subdivision of the time interval $[0, T]$ in M intervals $[T_\alpha, T_{\alpha+1}]$ of size $\varepsilon^k (n + 1)^{-1/2}$ as in the previous paragraph. As the same bound on the velocities applies, we may again assume that t_0 is one of the T_α in the following sense:

Lemma 4.7 *There exist i, j, α such that $i R_{T_\alpha} j$, and $\varepsilon^k < |X_i(T_\alpha) - X_j(T_\alpha)| \leq 2\varepsilon^k$.*

Proof We use the definition of $\tilde{B}_2$ and take α such that $T_\alpha = \sup_\beta \{T_\beta < t_0\}$. Neither i nor j may have any collision with another particle between T_α and t_0. Indeed, suppose the last collision of i between T_α and t_0 is with particle k, and ends at t_1. Between t_1 and t_0, the velocities of i and j are bounded by $\sqrt{n+1}$ thanks to Lemma 4.4. Thus, at t_1 i, j and k

would be in a ball of size $2\varepsilon^k$ and the initial condition would belong to $\tilde{B}_1$ and thus not $\tilde{B}_2$. Note now that since the trajectories of i and j are lines between two collisions, they cannot cross more than once. This implies that i and j do not collide between T_α and t_0. Thus, applying the velocity bound of Lemma 4.4, we have $\varepsilon^k < |X_i(T_\alpha) - X_j(T_\alpha)| \leq 2\varepsilon^k$ and $i \, R_{T_\alpha} \, j$ (remember that $i \, R_{t_0} \, j$). $\qquad\square$

Lemma 4.8 *For any $\beta \geq 1$ and any two points x_i and x_f in $[-1 - 2T, 1 + 2T]^3$, denote by $S(x_i, x_f, \beta)$ the set of initial conditions $X_1(0), X_2(0), V_1(0), V_2(0)$ in $[-1 - 2T, 1 + 2T]^6 \times [-2, 2]^6$ such that*

(i) $X_1(0) \in B(x_i, 2\varepsilon^k)$,
(ii) *the two particles 1 and 2 have a collision between time 0 and T_β (and no collision with any other particles), and*
(iii) $X_1(T_\beta) \in B(x_f, 2\varepsilon^k)$ or $X_2(T_\beta) \in B(x_f, 2\varepsilon^k)$.

Then

$$|S(x_i, x_f, \beta)| \leq C \frac{\varepsilon^{6k}}{\beta^2}.$$

Note that as the particles have no collision with other particles, the dynamics really involves only them, which is why S contains only the initial conditions for those two particles.

Proof The dynamics to consider is simply

$$\dot{X}_1(t) = X_1(t), \qquad \dot{X}_2(t) = V_2(t),$$

$$\dot{V}_1(t) = \frac{1}{N} K_\varepsilon(X_1 - X_2), \qquad \dot{V}_2 = \frac{1}{N} K_\varepsilon(X_2 - X_1).$$

Let us consider $S_j(x_i, x_f, \beta)$ for $j = 1, 2$ the subset of $S(x_i, x_f, \beta)$ for which $X_j(T_\beta) \in B(x_f, 2\varepsilon^k)$ (*i.e.* (iii) is true for X_j). The set S is the union of S_1 and S_2 and it is obviously enough to prove the estimate for those subsets.

Now denote by $\tilde{S}_j(x_i, x_f, \beta)$ the set of initial conditions $X_1(0), X_2(0), V_1(0), V_2(0)$ in $[-1 - 2T, 1 + 2T]^6 \times [-3, 3]^6$ such that again conditions (i)–(iii) of (4.8) are satisfied with $X_j(T_\beta) \in B(x_f, 2\varepsilon^k)$. The only difference between $\tilde{S}_j$ and S_j is that $\tilde{S}_j$ allows for a larger support in velocity initially.

For any fixed x_f and V with $|V| \leq 1$, we have that

$$S_j(x_i, x_f, \beta) + (0, 0, V, V) \subset \tilde{S}_j(x_i, x_f + VT_\beta).$$

This is indeed only Galilean invariance: if we add V to the velocities of each particle initially, then they follow the same dynamics with the same V added to their velocities and Vt added to their position. Hence the final position $X_j(T_\beta)$ is simply shifted by VT_β.

As a consequence for any x_f and any $y \in B(x_f, T_\beta)$, we have that

$$|\tilde{S}_j(x_i, y, \beta)| \geq |S_j(x_i, x_f, \beta)|.$$

Take L points y^n in the ball $B(x_f, T_\beta)$ such that the distance between any two of them is at least $4\varepsilon^k$. Then clearly

$$\tilde{S}_j(x_i, y^n, \beta) \cap \tilde{S}_j(x_i, y^p, \beta) = \emptyset, \quad \text{if } n \neq p,$$

Fig. 1 An example of collision sequence when particle i has no collision between T_β and T_α; the collision sequence of particles j and q may be more complicated than what is sketched here, in the limit of four or less collisions before T_α
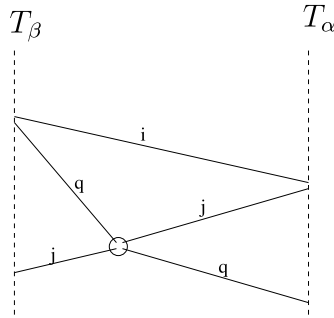
as the final position $X_j(T_\beta)$ cannot be in both $B(y^n, 2\varepsilon^k)$ and $B(y^p, 2\varepsilon^k)$. Obviously there are at most $L = C\beta^3$ such points y^n (remember that T_β is of the order of $\beta\varepsilon^k$).

On the other hand $U = \bigcup_{n=1}^{L} \tilde{S}_j(x_i, y^n, \beta)$ contains only those initial data such that $X_1(0) \in B(x_i, 2\varepsilon^k)$ and 1 and 2 have a collision between time 0 and T_β. Therefore its measure is at most $C\beta\varepsilon^{6k}$. Consequently

$$L \times |S_j(x_i, x_f, \beta)| \le \sum_{n=1}^{L} |\tilde{S}_j(x_i, y^n, \beta)| = \left| \bigcup_{n=1}^{L} \tilde{S}_j(x_i, y^n, \beta) \right| \le C\beta\varepsilon^{6k}.$$

So finally, we obtain that

$$|S_j(x_i, x_f, \beta)| \le C \frac{\varepsilon^{6k}\beta}{L} = C \frac{\varepsilon^{6k}}{\beta^2}. \qquad \square$$

As i and j belong to the same sequence before T_α, there exist two particles p and q (one or both of which could be i or j) colliding at some time $t_1 < T_\alpha$ (t_1 being here the final time of the collision) and such that p and i belong to the same sequence between t_1 and T_α and so do respectively q and j.

One of these sequences of collisions contains at least one collision: If not i and j would have one collision at t_1, another one at t_0 and none in between, which was already excluded.

On the other hand, one also has at most one collision. Indeed there are at most four collisions before T_α and one already occurred at t_1 so there are only three remaining. This is too few to have at least two other collisions for each sequence.

Define T_β as the smallest of the T_γ bigger than t_1. p and q have at most 3 collisions between t_1 and T_β, and the $\sqrt{n+1}$ velocity bound applies between the collision, so that

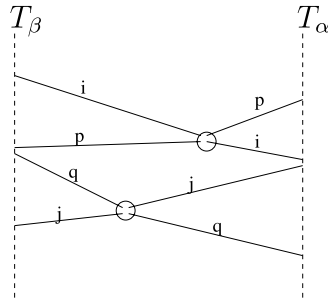$$|X_p(T_\beta) - X_q(T_\beta)| \le 6\varepsilon^k,$$

and we may replace t_1 by T_β.

Now assume that the sequence with p and i between t_β and T_α is the one with the less collisions (and hence 0 or 1). Let us consider both cases

No collision for i in $[T_\beta, T_\alpha]$ That means that $i = p$. Accordingly we know that $j \ne q$. Let us denote by $\tilde{B}_2^1$ the set of initial data such that this occurs and estimate its measure. Figure 1 gives an example of collision sequence corresponding to this case.

In order to do so, we bound the set of positions and velocities at time T_β and use Liouville's theorem to deduce a bound on $\tilde{B}_2^1$. Therefore denote by $C_1(\beta)$ the set of corresponding

Fig. 2 An example of collision sequence when particle *i* has one collision between T_β and T_α; the collision sequence of particles *j* and *q* may be more complicated than what is sketched here, in the limit of four or less collisions before T_α

positions and velocities st time T_β $X_n(T_\beta)$, $V_n(T_\beta)$, $1 \leq n \leq N$. We have that

$$|\tilde{B}_2^1| \leq \sum_\beta |C_1(\beta)|.$$

First choose $X_n(T_\beta)$, $V_n(T_\beta)$ for any $n \neq i$ such that *j* and *q* belong to the same sequence of collisions between T_β and T_α (of any number of collisions); Denote by $C(j, q, i, \beta, \alpha)$ the corresponding set. The positions and velocities of all particles except *i* are known between T_β and T_α, as *i* does not interact with any of them within this time interval.

Once this is done, we must choose $X_i(T_\beta)$ in the ball $B(X_q(T_\beta), \varepsilon^k)$ and as its trajectory is a line, $V_i(T_\beta)$ must lie in the ball centered at $(X_j(T_\alpha) - X_i(T_\beta))/(T_\alpha - T_\beta)$, and of radius $\varepsilon^k/(T_\alpha - T_\beta))$. Therefore the set of corresponding $X_i(T_\beta)$ and $V_i(T_\beta)$ is of measure less than $C\varepsilon^{3k}(\alpha - \beta)^{-3}$. Moreover

$$|C_1(\beta)| \leq \sum_{i,j,q} \sum_\alpha \int_{C(j,q,i,\beta,\alpha)} \int_{X_i \in B(X_q, \varepsilon^k)}$$

$$\times \int_{(T_\alpha - T_\beta)V_i \in B((X_j(T_\alpha) - X_i), \varepsilon^k)} dV_i dX_i dX_1 dV_1 \ldots dX_N dV_N$$

$$\leq C \sum_{i,j,q} \sum_\alpha \varepsilon^{6k} (T_\alpha - T_\beta)^{-3} |C(j, q, i, \beta, \alpha)|.$$

The last step is to evaluate the measure of $C(j, q, i, \beta, \alpha)$. As there is at least one collision between T_β and T_α, say between particles *k* and *l*, there exists $\gamma \in [\beta, \alpha]$ such that *k* and *l* are in the same ball of size $2\varepsilon^k$. Evaluating the measure of $C(j, q, i, \beta, \alpha)$ by its image by the flow at time T_γ, and summing over all possible γ, we obtain that $|C(j, q, i, \beta, \alpha)| \leq C\varepsilon^{3k}(\alpha - \beta)$. Finally, we conclude:

$$|C_1(\beta)| \leq CN^3 \varepsilon^{6k},$$

and

$$|\tilde{B}_2^1| \leq \sum_\beta |C_1(\beta)| \leq CN^3 \varepsilon^{5k},$$

which is the desired estimate.

One collision for i in $[T_\beta, T_\alpha]$ That means that $p = i$ or $p = l$. Figure 2 gives an example of collision sequence corresponding to this case.

We follow the same steps as in the previous lemma, and denote by $\tilde{B}_2^2$ the corresponding set of initial conditions and $C_2(\beta)$ the set of $X_n(T_\beta)$, $V_n(T_\beta)$.

As before we first choose $X_n(T_\beta)$, $V_n(T_\beta)$ for $n \neq i, l$ in the set $C(j, q, i, l, \beta, \alpha)$, which is such that j and q belong to the same sequence of collisions between T_β and T_α. This set is almost exactly $C(j, q, i, \beta, \alpha)$ except that particle l is not included.

Then it remains to choose $X_i(T_\beta)$, $V_i(T_\beta)$, $X_l(T_\beta)$ and $V_l(T_\beta)$ such that i and l have a collision between T_β and T_α, $X_i(T_\alpha)$ belongs to $B(X_j(T_\alpha), 2\varepsilon^k)$ and $X_i(T_\beta)$ or $X_l(T_\beta)$ belong to $B(X_q(T_\beta), 2\varepsilon^k)$. This is the set $S(X_q(\beta), X_j(\alpha), \alpha - \beta)$ of Lemma 4.8, with the roles of x_i and x_f exchanged. As the dynamics is reversible, Lemma 4.8 applies. So

$$|C_2(\beta)| \leq \sum_{i,j,q} \sum_\alpha \int_{C(j,q,i,l,\beta,\alpha)} \int_{S(X_q, X_j(\beta), \alpha-\beta)} dV_i dX_i dV_l dX_l dX_1 dV_1 \ldots dX_N dV_N$$

$$\leq C \sum_{i,j,q,l} \sum_\alpha \varepsilon^{6k}(\alpha-\beta)^{-2} |C(j,q,i,l\beta,\alpha)|,$$

according to Lemma 4.8. The estimate for $|C(j, q, i, l, \beta, \alpha)|$ being the same as before, one obtains

$$|\tilde{B}_2^2| \leq C \sum_{\alpha>\beta} \sum_{i,j,q,l} \varepsilon^{9k}(\alpha-\beta)^{-1} \leq C N^4 \varepsilon^{8k} |\ln \varepsilon|,$$

which is asymptotically less than $N^3 \varepsilon^{5k}$ as $N\varepsilon^{3k}|\ln \varepsilon|$ converges to 0 ($N = \varepsilon^{-6}$ and $k > 3$). Finally this does imply that

$$|\tilde{B}_2| \leq |\tilde{B}_2^1| + |\tilde{B}_2^2| \leq C N^3 \varepsilon^{5k}.$$

References

1. Ambrosio, L.: Transport equation and Cauchy problem for BV vector fields. Invent. Math. **158**(2), 227–260 (2004)
2. Braun, W., Hepp, K.: The Vlasov dynamics and its fluctuations in the $1/N$ limit of interacting particles. Commun. Math. Phys. **56**, 101–113 (1977)
3. Cercignani, C., Illner, R., Pulvirenti, M.: The Mathematical Theory of Dilute Gases. Applied Mathematical Sciences, vol. 106. Springer, New York (1994)
4. Dobrushin, R.L.: Vlasov equations. Funct. Anal. Appl. **13**, 115–123 (1979)
5. Goodman, J., Hou, T.Y., Lowengrub, J.: Convergence of the point vortex method for the 2-D Euler equations. Commun. Pure Appl. Math. **43**, 415–430 (1990)
6. Hauray, M.: On Liouville transport equation with force field in BV_{loc}. Commun. Partial Differ. Equ. **29**(1–2), 207–217 (2004)
7. Hauray, M., Jabin, P.E.: N-particles approximation of the Vlasov equation with singular potential. Arch. Ration. Mech. Anal. **183**(3), 489–524 (2007)
8. Illner, R., Pulvirenti, M.: Global validity of the Boltzmann equation for two and three-dimensional rare gas in vacuum. Commun. Math. Phys. **121**, 143–146 (1989)
9. Jabin, P.E., Otto, F.: Identification of the dilute regime in particle sedimentation. Commun. Math. Phys. **250**, 415–432 (2004)
10. Lanford, O.E.: In: Dynamical Systems, Theory and Applications, Battelle Rencontres, Seattle, WA, 1974. Lecture Notes in Phys., vol. 38, pp. 1–111. Springer, Berlin (1975)
11. Lanford, O.E.: On a derivation of the Boltzmann equation. In: International Conference on Dynamical Systems in Mathematical Physics, Rennes, 1975. Asterisque, vol. 40, pp. 117–137. Soc. Math. France, Paris (1976)
12. Neunzert, H., Wick, J.: Theoretische und numerische Ergebnisse zur nicht linearen Vlasov Gleichung. In: Numerische Lösung nichtlinearer partieller Differential und Integrodifferentialgleichungen, Tagung, Math. Forschungsinst., Oberwolfach, 1971. Lecture Notes in Math., vol. 267, pp. 159–185. Springer, Berlin (1972)

13. Schochet, S.: The weak vorticity formulation of the 2-D Euler equations and concentration-cancellation. Commun. Partial Differ. Equ. **20**, 1077–1104 (1995)
14. Schochet, S.: The point-vortex method for periodic weak solutions of the 2-D Euler equations. Commun. Pure Appl. Math. **49**, 911–965 (1996)
15. Spohn, H.: Large Scale Dynamics of Interacting Particles. Springer, Berlin (1991)
16. Victory, H.D. Jr., Allen, E.J.: The convergence theory of particle-in-cell methods for multidimensional Vlasov-Poisson systems. SIAM J. Numer. Anal. **28**, 1207–1241 (1991)
17. Wollman, S.: On the approximation of the Vlasov-Poisson system by particles methods. SIAM J. Numer. Anal. **37**, 1369–1398 (2000)